Lefschetz-thimble Integral & SSB of Matrix Models

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Introduction and Motivation

Motivation

Path integral with **complex weights** appear in many important physics:

- Finite-density lattice QCD,
  spin-imbalanced nonrelativistic fermions
- Gauge theories with topological θ terms
- Real-time quantum mechanics

Oscillatory nature **hides** many important properties of partition functions.

Example: Airy integral

Let's consider a one-dimensional oscillatory integration:

$$\mathrm{Ai}(a)=\int_{\mathbb{R}}\frac{\mathrm{d}x}{2\pi}\exp\mathrm{i}\left(\frac{x^3}{3}+ax\right).$$

RHS is well defined **only if** $\mathrm{Im}\,a=0$, though $\mathrm{Ai}(z)$ is **holomorphic**.

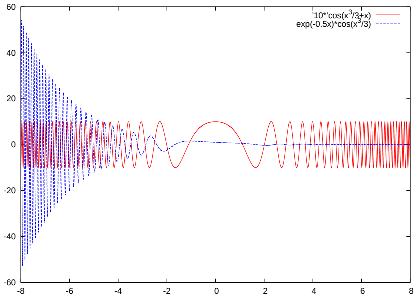

Figure : Real parts of integrands for $a=1$ $(\times 10)$ & $a=0.5\mathrm{i}$

Contents

- How can we circumvent such oscillatory integrations?
  $\Rightarrow$ **Lefschetz-thimble integrations**
  [Witten, arXiv:1001.2933, 1009.6032]

- Applications of this new technique for path integrals
  - Phase transition of the matrix model
    [YT, PRD91,036002; Kanazawa, YT, arXiv:1412.2802]

Introduction to Lefschetz-thimble integrations

Lefschetz-thimble method = Steepest descent integration

Oscillatory integrals with **many variables** can be evaluated using the "'steepest descent" cycles $\mathcal{J}_\sigma$:

$$\int_{\mathbb{R}^n} \mathrm{d}^n x\, e^{\mathrm{i}S(x)} = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathrm{d}^n z\, e^{\mathrm{i}S(z)}.$$

$\mathcal{J}_\sigma$ are called Lefschetz thimbles, and $\mathrm{Im}[\mathrm{i}S]$ is constant on it.

n_σ: intersection numbers of duals $\mathcal{K}_\sigma$ and $\mathbb{R}^n$.

Example: Airy integral

Airy integral:

$$\mathrm{Ai}(a) = \int_{\mathbb{R}} \frac{\mathrm{d}x}{2\pi} \exp \mathrm{i} \left(\frac{x^3}{3} + ax \right).$$

The integrand is **holomorphic** w.r.t x

$\Rightarrow$ The contour can be deformed continuously without changing the value of the integration!

Steepest descent contours

What is the most appropriate contour for our purpose?

$\mathrm{Re}[\mathrm{i}S(x,a)]$ should be made as small as possible.
$\Leftrightarrow$ The contour should be **perpendicular** to $\mathrm{Re}[\mathrm{i}S(x,a)] = \mathrm{cosnt.}$

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Steepest descent ones $\mathcal{J}$ must satisfy $\mathrm{Im}[\mathrm{i}S(x,a)] = \mathrm{const.}$
because of the holomorphy.

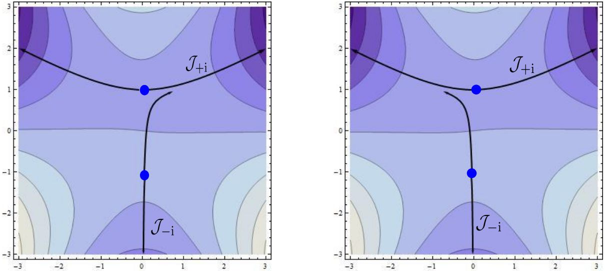

Figure : Contour plots for $\mathrm{Re}[\mathrm{i}S(x,a)]$ with $a = \exp \pm \mathrm{i}0.1$.

Rewrite the Airy integral

There exists two Lefschetz thimbles $\mathcal{J}_\sigma$ $(\sigma = 1, 2)$ for the Airy integral:

$$\mathrm{Ai}(a) = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \frac{\mathrm{d}z}{2\pi} \exp \mathrm{i} \left(\frac{z^3}{3} + az \right).$$

n_σ: intersection number of the steepest ascent contour $\mathcal{K}_\sigma$ and $\mathbb{R}$.

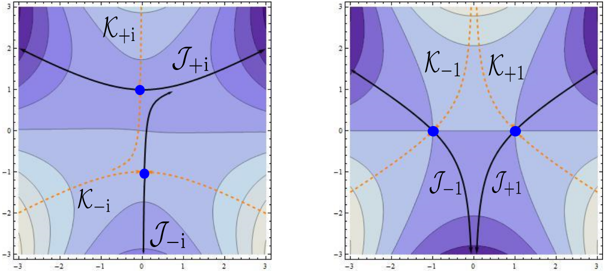

Figure : Lefschetz thimbles $\mathcal{J}$ and duals $\mathcal{K}$ $(a = \exp(0.1\mathrm{i}),\ \exp(\pi \mathrm{i}))$

Generalization to multiple integrals

Model integral:

$$Z = \int_{\mathbb{R}^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \exp \mathcal{I}(x_i).$$

What properties are required for Lefschetz thimbles $\mathcal{J}$?

1. $\mathcal{J}$ should be a n-dimensional object in $\mathbb{C}^n$.
2. $\mathrm{Im}[\mathcal{I}]$ should be constant on $\mathcal{J}$.

Short note on technical aspects

Complexified variables ($a = 1, \ldots, n$): $z_a = x_a + \mathrm{i} p_a$.
Regard x_a as **coordinates** and p_a as **momenta**, so that **Poisson bracket** is given by

$$\{f, g\} = \sum_{a=1,2} \left[\frac{\partial f}{\partial x_a} \frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial x_a} \frac{\partial f}{\partial p_a} \right].$$

Short note on technical aspects

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Hamilton equation with the Hamiltonian $H=\mathrm{Im}[\mathcal{I}(z_a)]$:

$$\frac{\mathrm{d}f(x,p)}{\mathrm{d}t}=\{H,f\}\qquad\left(\Leftrightarrow\frac{\mathrm{d}z_a}{\mathrm{d}t}=-\overline{\left(\frac{\partial\mathcal{I}}{\partial z_a}\right)}\right)$$

This is **Morse's flow equation**: $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}[\mathcal{I}]\le 0$.
$\Rightarrow$ We can find n good directions for $\mathcal{J}$ around saddle points! [Witten, 2010]

Phase transition associated with symmetry

0-dim. Gross–Neveu-like model

The partition function of our model study is the following:

$$Z_N(G,m)=\int \mathrm{d}\bar\psi \mathrm{d}\psi\ \exp\left\{\sum_{a=1}^{N}\bar\psi_a(i\not{p}+m)\psi_a+\frac{G}{4N}\left(\sum_{a=1}^{N}\bar\psi_a\psi_a\right)^2\right\}.$$

The Hubbard–Stratonovich transformation gives

$$Z_N(G,m)=\sqrt{\frac{N}{\pi G}}\int_{\mathbb{R}}\mathrm{d}\sigma\,\mathrm{e}^{-NS(\sigma)},$$

with

$$S(\sigma)\equiv\frac{\sigma^2}{G}-\log[p^2+(\sigma+m)^2].$$

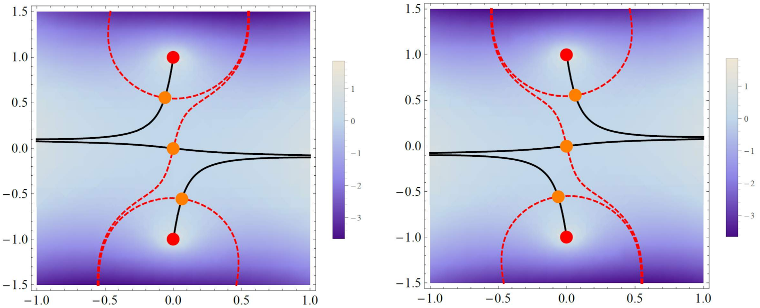
For simplicity, we put $m=0$ in the following.

Properties of $S(\sigma)$

S has three saddle points:

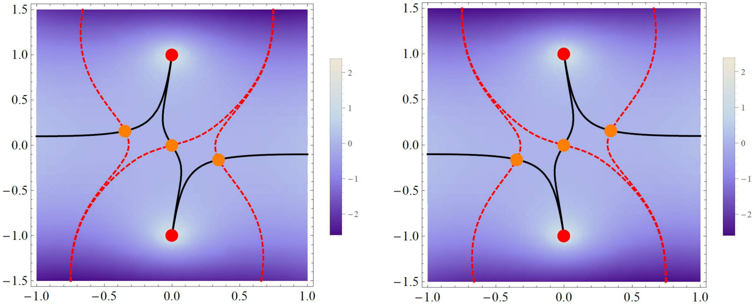
$$0=\frac{\partial S(z)}{\partial z}=\frac{2z}{G}-\frac{2z}{p^2+z^2}\quad\Longrightarrow\quad z=0,\ \pm\sqrt{G-p^2}\,.$$

Figures for $G=0.7e^{\mp 0.1\mathrm{i}}$ and $p^2=1$ [Kanazawa, YT, arXiv:1412.2802]:



Properties of $S(\sigma)$

Figures for $G = 1.1e^{\mp 0.1\mathrm{i}}$ and $p^2 = 1$ [Kanazawa, YT, arXiv:1412.2802]:



From these figures, we learn that, for real G,

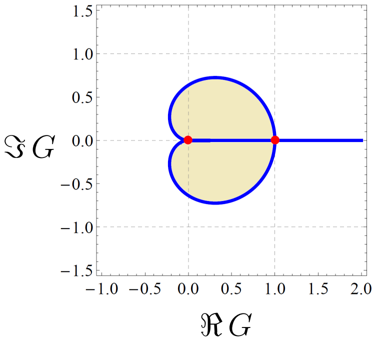
- $z = 0$ is the unique critical point contributing to Z if $G < p^2$.
- All three critical points contribute to Z if $G > p^2$.

Stokes phenomenon

The difference of the way of contribution can be described by **Stokes phenomenon**.

$\Leftarrow$ At some **special values** of coupling, several critical points are connected by the flow. [Witten, arXiv:1001.2933, 1009.6032]

Figures of G-plane for $\mathrm{Im}S(0) = \mathrm{Im}S(z_\pm)$ [Kanazawa, YT, arXiv:1412.2802]:



Dominance of contribution

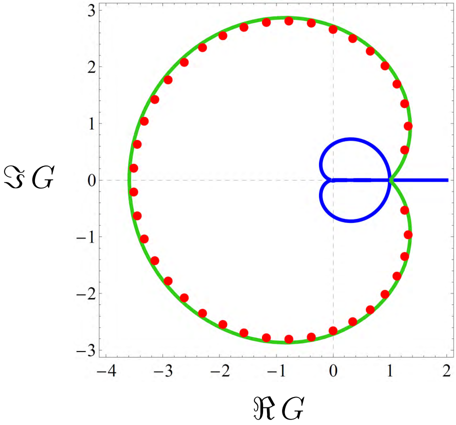
The Stokes phenomenon tells us the **number** of Lefschetz thimbles contributing to Z_N.
However, it does **not** tell which thimbles give main contribution.

$$Z_N \sim \#\exp(-NS(0)) + \#\exp(-NS(z_\pm))$$

In order to obtain $\langle\sigma\rangle \neq 0$ in the large-N limit, $z_\pm$ should dominate $z = 0$.

$$\Rightarrow \qquad \mathrm{Re}S(z_\pm) \leq \mathrm{Re}S(0)$$

Connection with Lee–Yang zero



Blue line: $\mathrm{Im}S(z_\pm) = \mathrm{Im}S(0)$.
Green line: $\mathrm{Re}S(z_\pm) = \mathrm{Re}S(0)$.
Red points: Lee-Yang zeros at $N = 40$. [Kanazawa, YT, arXiv:1412.2802]

Conclusions for studies with GN-like models

1. Decomposition of the integration path in terms of Lefschetz thimbles is useful to visualize different phases.
2. The possible link between thimble decomposition and Lee–Yang zeros is indicated.